Tradeoffs for Downside Risk Averse Decision Makers and the Self-Protection Decision

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Abstract: Besides risk aversion, decision makers are often assumed to be downside risk averse. In order to investigate tradeoffs that downside risk averse decision makers face, this paper proposes five stochastic orders, each corresponding to a tradeoff involving a downside risk increase. In addition to obtaining their respective CDF characterizations, these orders are also combined with Ross more risk aversion and two versions of Ross more downside risk aversion to produce comparative static theorems identifying the choices of decision makers relative to that of a reference decision maker. The paper concludes with analysis of the decision to self-protect, a decision that increases downside risk along with making other changes. This exercise not only shows that all five stochastic orders studied in this paper find corresponding tradeoffs in the self-protection model, it also demonstrates that these five tradeoffs are the only meaningful tradeoffs that the standard self-protection model creates. Therefore, the concepts and results presented here provide a systematic and complete treatment of the relationship between self-protection and risk preferences.

Key Words: risk, downside risk, risk aversion, prudence, self-protection
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1. Introduction

Decision making in economics is about tradeoffs. For risk averse decision makers the most important tradeoff is between the size of a random variable and its riskiness. This particular tradeoff has been discussed for many years, often in mean-variance decision models, and recently in the expected utility setting as well.[1] When only $u'(x) \geq 0$ and $u''(x) \leq 0$ are assumed for utility function $u(x)$, this tradeoff of size for risk is the only one that can be discussed. Recently, it has become common to add the assumption of downside risk aversion or prudence, a property characterized by $u'''(x) \geq 0$.[2] The focus of the analysis here is on the tradeoffs that are made possible by this additional assumption. These are the tradeoffs that decision makers who are downside risk averse face when choosing among random variables.

Whenever a random variable is altered, the change that occurs is either beneficial or harmful depending on the risk preferences of the decision maker. In expected utility terms, the change either increases or decreases expected utility. When two such changes are made, and these changes offset one another for a reference decision maker, information concerning that decision maker's willingness to trade off the one change for the other is revealed, and this information can be used to infer the choices that would be made by other decision makers whose risk preferences differ from the reference person in some specific way. In order to investigate the tradeoffs facing downside risk averse decision makers, this paper introduces five stochastic orders, each corresponding to a tradeoff involving a downside risk increase.

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[1] Liu and Meyer (2013, 2015).

[2] Downside risk aversion is the term used by Menezes, Geiss and Tressler (1980). Kimball (1990) uses the term prudence for the risk attitude captured by $u'''(x) \geq 0$.

There are two major components or steps in the analysis, and each leads to important theoretical results. The first step determines the condition on a pair of cumulative distribution functions (CDF) that reflects the fact that two specified changes to the random variable have occurred. It is the case that this first step results in a stochastic dominance theorem, or equivalently, defines and characterizes a partial order over random variables.

As an example where this step has been carried out before, consider second degree stochastic dominance (SSD). When a random variable given by CDF G(x) is made larger and then less risky resulting in F(x), the condition on [G(x) - F(x)] that is implied by these two changes is precisely the well known second degree stochastic dominance condition. In the mathematical statistics literature, this SSD condition is said to characterize the increasing concave order. Existing theory in both the economics and mathematical statistics literatures has presented many such CDF characterizations associated with either a form of stochastic dominance, or a partial order over random variables.[3] The work presented here adds several more results of this type even though that is not its primary objective.

The second step in the analysis is the one that accomplishes the primary goal of the research, predicting the choices of decision makers. This step takes the information generated by observing the ranking of two random variables by a reference decision maker and uses it to determine how those same two alternatives would be ranked by others. That is, theorems are presented that indicate which decision makers, defined relative to a reference decision maker, would choose F(x) over G(x) whenever the reference decision maker does so.

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[3] Ekern (1980), for instance, presents definitions of nth degree stochastic dominance and nth degree increases in risk.

Theorems with this structure have been presented before. Diamond and Stiglitz (1974), for instance, define a mean utility preserving spread as a particular change to a random variable that leaves a reference decision maker indifferent between the initial and changed random variable. The CDF condition characterizing this definition is then used to show that all decision makers who are Arrow-Pratt more risk averse than the reference agent would not choose the mean utility preserving spread. The comparative static theorems demonstrated in Section 3 follow this same pattern.

To discuss the tradeoff of an increase in downside risk – which decreases the expected utility when the decision maker is downside risk averse – for some other change in the random variable, it must be that this other change increases expected utility. For decision makers whose utility function satisfies $u'(x) \geq 0$, $u''(x) \leq 0$ and $u'''(x) \geq 0$, there are five possible ways to increase expected utility to offset an increase in downside risk. For example, expected utility is increased whenever the risk of the random variable is decreased. Decision makers with the stated risk aversion properties unanimously agree that the decrease in risk is beneficial, and that the increase in downside risk is harmful. Other changes that can offset an increase in downside risk include an increase in size (in the sense of an FSD change) and the various combinations of increases in size and decreases in risk. There are a total of five such possibilities, and these are fully described and discussed in the various subsections of Section 3.

Using self-protection as an example, the paper demonstrates the applicability of the concepts and results presented here. Not only do all five stochastic orders find corresponding tradeoffs in the self-protection model, it is also the case that these five tradeoffs are the only meaningful tradeoffs that the standard self-protection model creates. The analysis confirms the findings in the literature that more downside risk averse individuals tend to invest less in self-protection, and it demonstrates this point in a more systematic and complete fashion.

The paper is organized as follows. First, in Section 2, several well known changes to random variables are reviewed and the notation and assumptions of the paper are established. This review covers changes to random variables that can compensate for or offset an increase in downside risk. Included with this review is discussion of extensions of Ross's strongly more risk averse order over decision makers. After this review, the primary analysis of the paper is presented in Section 3, which is divided into five subsections. Each subsection considers a different tradeoff that a downside risk averse decision maker can face. Within each subsection are two theorems. The first theorem characterizes the stochastic order that is necessary for the random variable to have undergone a downside risk increase and an accompanying change that increases expected utility. Using risk aversion properties, the second theorem in each subsection identifies decision makers who prefer the two changes and those who do not. These sets of decision makers are defined relative to a reference decision maker who is indifferent. The extended versions of Ross's strongly more risk averse order are used to accomplish this. In the first subsection, extensive discussion and intuition is provided for each of the two theorems. Having done this, much less discussion is required in the remaining four subsections. Section 4 applies the theoretical findings from Section 3 to the self-protection decision model.

2. Preliminaries and Literature Review

Determining when a change in a random variable increases expected utility for broad groups of decision makers has been and still is a major area of research in decision making under risk. The term stochastic dominance is used to describe much of this research, and first and second degree stochastic dominance are prominent examples. The definitions of increased risk and increased downside risk also fit this description. After reviewing these four changes and the CDF characterization for each, a fifth and less familiar change is also reviewed. This change to a random variable defines the increasing convex order. An extension of Ross's strongly more risk averse order is also reviewed. All of this is done to establish the building blocks for the analysis in Section 3.

In terms of notation, let $\tilde{x}$ and $\tilde{y}$ denote two random variables with cumulative distribution functions F(x) and G(x) respectively. Assume that the supports of all random variables lie in a bounded interval denoted [a, b] with no probability mass at the left endpoint a. This implies that F(a) = G(a) = 0. Of course, for all cumulative distribution functions with support in [a, b], F(b) = G(b) = 1. It is assumed that expected utility is maximized and that any utility function u(x) is differentiable at least three times. In addition, let μ_F and μ_G denote the mean values of these alternatives, and σ_F^2 and σ_G^2 their variances.

Hadar and Russell (1969) and Hanoch and Levy (1969) define both a first degree stochastic dominant (FSD) change and a second degree stochastic dominant (SSD) change.

<u>Definition 1</u>: $\tilde{x}$ dominates $\tilde{y}$ in the first degree if $E_F u(x) \geq E_G u(x)$ for all u(x) with $u'(x) \geq 0$.

<u>Definition 2</u>: $\tilde{x}$ dominates $\tilde{y}$ in the second degree if $E_F u(x) \geq E_G u(x)$ for all u(x) with $u'(x) \geq 0$ and $u''(x) \leq 0$.

It is well known that F(x) dominates G(x) in FSD if and only if $G(x) \geq F(x)$ for all x in [a, b], and that F(x) dominates G(x) in SSD if and only if $\int_a^y [G(x) - F(x)]dx \geq 0$ for all y in [a, b].

An increase in risk is defined by Rothschild and Stiglitz (R-S) (1970), and an increase in downside risk by Menezes, Geiss and Tressler (MGT) (1980).

Definition 3: $\tilde{y}$ is riskier than $\tilde{x}$ if $E_F u(x) \geq E_G u(x)$ for all u(x) with $u''(x) \leq 0$.

Definition 4: $\tilde{y}$ is downside riskier than $\tilde{x}$ if $E_F u(x) \geq E_G u(x)$ for all u(x) with $u'''(x) \geq 0$.

R-S show that G(x) is riskier than F(x) if and only if $\int_a^y [G(x) - F(x)]dx \geq 0$ for all y in [a, b] with equality holding at y = b. For downside risk increases, MGT show that G(x) is downside riskier than F(x) if and only if $\int_a^b [G(x) - F(x)]dx = 0$ and $\int_a^y \int_a^x [G(s) - F(s)]dsdx \geq 0$ for all y in [a, b] with equality holding at y = b. A downside increase in risk implies that $\mu_F = \mu_G$, and $\sigma_F^2 = \sigma_G^2$.

The mathematical statistics literature describes these same four concepts using different terminology. For instance, an FSD or SSD change is described as a change where $\tilde{x}$ is larger than $\tilde{y}$ in the increasing order, or in the increasing concave order, respectively. Similarly, $\tilde{y}$ is an increase in risk from $\tilde{x}$ is equivalent to $\tilde{x}$ being larger than $\tilde{y}$ in the concave order. The change to a random variable that is reviewed next uses similar terminology.

Definition 5: Random variable $\tilde{y}$ is larger than $\tilde{x}$ in the increasing convex order if $E_G[u(x)] \geq E_F[u(x)]$ for all u(x) with $u'(x) \geq 0$ and $u''(x) \geq 0$.

The CDF condition that characterizes this order is $\int_a^y [G(x) - F(x)]dx \geq -Q$ for all y in [a, b] where $Q = \mu_G - \mu_F \geq 0$.[4]

To identify those decision makers who dislike downside risk increases more than a reference decision maker, the Ross more risk averse definition is extended to the third degree. This follows the nth degree generalization by Liu and Meyer (2013).[5] As with Ross's definition, the extensions can be characterized in several different ways. Two characterizations are given for each of the following two definitions. For these definitions, it is assumed that u(x) and v(x) have positive first derivative, negative second derivative and positive third derivative on [a, b].

<u>Definition 6</u>: u(x) is (3/2)rd degree Ross more risk averse than v(x) on [a, b], if

i) there exists a $\lambda > 0$ such that $\dfrac{u'''(x)}{v'''(x)} \geq \lambda \geq \dfrac{u''(y)}{v''(y)}$ for all x and y in [a, b];

or equivalently, if

ii) there exists a $\lambda > 0$ and a function $\phi(x)$ with $\phi''(x) \geq 0$ and $\phi'''(x) \geq 0$ such that $u(x) = \lambda v(x) + \phi(x)$ for all x in [a, b].

<u>Definition 7</u>: u(x) is (3/1)rd degree Ross more risk averse than v(x) on [a, b] if

i) There exists a $\lambda > 0$ such that $\dfrac{u'''(x)}{v'''(x)} \geq \lambda \geq \dfrac{u'(y)}{v'(y)}$ for all x and y in [a, b];

[4] A definition of larger in the increasing convex order is available in Shaked and Shanthikumar (2007). A related concept, the "stop-loss order" is found in the actuarial science literature (Denuit et al. 2005).

[5] Definition 7 below is the Ross more downside risk aversion first defined in Modica and Scarsini (2005). Higher degree extensions of Definition 7 can be found in Jindapon and Neilson (2007), Li (2009), Denuit and Eeckhoudt (2010a), and Liu and Meyer (2013). Third degree Ross more risk aversion in Definition 6 is based on Liu and Meyer (2013).

or equivalently, if

ii) There exists a $\lambda > 0$ and a function $\phi(x)$ with $\phi'(x) \leq 0$ and $\phi'''(x) \geq 0$ such that $u(x) = \lambda v(x) + \phi(x)$ for all x in [a, b].

3. Tradeoffs for Downside Risk Averse Decision Makers

It is important to recognize that the decision makers whose choices are discussed throughout this research are those whose utility functions satisfy $u'(x) \geq 0$, $u''(x) \leq 0$ and $u'''(x) \geq 0$. These decision makers are risk averse, downside risk averse, and prefer larger outcomes. Even though these are the decision makers under consideration, when discussing tradeoffs for these decision makers, it is the case that the analysis procedure will require different assumptions for $u'(x)$ or $u''(x)$ or both. This occurs as part of the process of finding conditions on CDFs that are implied by an increase in downside risk and an increase in size or decrease in risk. In the first subsection, the beneficial change that is considered is a decrease in the riskiness of the random variable.

A. The Risk-Downside Risk Tradeoff

One tradeoff that a decision maker with utility function u(x) satisfying $u'(x) \geq 0$, $u''(x) \leq 0$ and $u'''(x) \geq 0$ can consider is whether or not to accept an increase in downside risk when it is accompanied by a decrease in risk. That is, total risk is reduced, but the risk that remains is more concentrated in the left tail. To establish notation, consider the following two changes to a random variable with CDF F(x). First, F(x) is changed to H(x) where H(x) is less risky than F(x) in the R-S sense. Next this H(x) is changed to G(x) where G(x) is downside riskier than H(x).[6] For the decision makers considered here, the first change increases and second change decreases expected utility, and this is precisely what is needed to make the discussion of tradeoffs possible.

The tool that is needed to continue the analysis is the condition on [G(x) - F(x)] that is implied by this risk decrease and downside risk increase. To find this condition, a different group of decision makers is considered, and a different but related question is posed and answered. This related question asks what condition on [G(x) - F(x)] is necessary and sufficient for all decision makers with $u''(x) \le 0$ and $u'''(x) \le 0$ to unanimously choose G(x) over F(x). The reason for asking this particular question is that its answer is also an answer to the original question. Decision makers in this group, with admittedly unusual risk preferences, would unanimously accept an increase in downside risk accompanied by a decrease in risk. This is because for this group of decision makers, both of these changes are beneficial. The logic being employed here is that a sufficient condition for G(x) to always be chosen over F(x) for these unusual decision makers is that G(x) can be obtained from F(x) by decreasing the risk and increasing the downside risk. As an aside, notice that no requirement on u'(x) is imposed. This is because preference for a risk decrease or a downside risk increase does not depend on the sign of u'(x).

The new finding in Theorem A1 that provides a necessary and sufficient condition on [G(x) - F(x)] for unanimous preference of G(x) over F(x) by those with $u''(x) \le 0$ and $u'''(x) \le 0$ could be called a stochastic dominance theorem. In choosing terminology for this paper, however, the convention used follows that established in the mathematical statistics literature,

[6] Which of the two changes occurs first does not alter the implied conditions on F(x) and G(x) or the other analysis presented here.

and the associated partial order over CDFs is named instead. Just as the term increasing convex order was used in Definition 5 to reflect the fact that u'(x) ≥ 0 and u"(x) ≥ 0 are assumed, the partial order defined in this subsection and characterized in Theorem A1 is called the concave imprudent order because u"(x) ≤ 0 and u'"(x) ≤ 0 are assumed.

Definition A: G(x) is larger than F(x) in the concave imprudent order if $E_G u(x) \geq E_F u(x)$ for all u(x) with u"(x) ≤ 0, and u'"(x) ≤ 0.[7]

The following theorem identifies a necessary and sufficient condition on F(x) and G(x) for them to be linked by the concave imprudent order. The proofs of Theorem A1 and all theorems that follow are in the Appendix. The notation used in these theorems considers the cumulative distribution function F(x) = $F^{[1]}$(x), and then denotes higher order cumulative functions using $F^{[k]}(x) = \int_a^x F^{[k-1]}(y)dy, \quad k = 2,3$. Similar notation applies to G(x) and other CDFs.

Theorem A1: G(x) is larger than F(x) in the concave imprudent order if and only if

$$G^{[2]}(b) - F^{[2]}(b) = 0$$
$$G^{[3]}(x) - F^{[3]}(x) \geq G^{[3]}(b) - F^{[3]}(b), \quad \forall x \in [a,b]$$

Theorem A1 provides the tool needed to complete the analysis of the tradeoff of risk for downside risk. Recall that the assumption made for all decision makers is that u'(x) ≥ 0, u"(x) ≤ 0 and u'"(x) ≥ 0. Therefore when G(x) is obtained from F(x) by reducing risk and

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[7] Because there is one definition and two theorems in each of the five subsections, they are labeled in a way that indicates their close association with one another.

increasing downside risk, some decision makers prefer G(x), some prefer F(x), and there are those who are indifferent between F(x) and G(x). That is, since one change is beneficial and the other is harmful, depending on the sizes of the two changes and on the sensitivity to risk and downside risk, the combined effect can be beneficial, harmful or neutral. For discussing tradeoffs this is perfect. When a reference decision maker indicates his choice between F(x) and G(x), information concerning that decision maker's willingness to trade off risk for downside risk is revealed, this is information that can then be used to infer the choices of others.

When G(x) is obtained from F(x) by reducing risk and increasing downside risk and a risk averse and downside risk averse decision maker is indifferent, one would expect that those decision makers who dislike risk more, or those who are less averse to downside risk, or both, would prefer G(x), while those whose preferences relative to the reference decision maker go in the opposite direction would prefer F(x). This is indeed the case, and formally stating this is the subject of Theorem A2. This theorem identifies groups of decision makers, defined relative to a reference decision maker, whose choice between F(x) and G(x) can be inferred from knowing the selection made by the reference decision maker.

<u>Theorem A2</u>: Suppose that G(x) is larger than F(x) in the concave imprudent order. Then

(a) $E_F v(x) \geq E_G v(x)$ implies $E_F u(x) \geq E_G u(x)$ for all u(x) who are (3/2)rd degree Ross more risk averse than v(x).

(b) $E_F v(x) \leq E_G v(x)$ implies $E_F u(x) \leq E_G u(x)$ for all u(x) who are (3/2)rd degree Ross less risk averse than v(x).

It is important to recognize that when F(x) is changed to H(x) where H(x) is less risky than F(x), and H(x) is changed to G(x) where G(x) is downside riskier than H(x), then G(x) is larger than F(x) in the concave imprudent order.[8] Thus, when any decision maker is indifferent between having or not having an increase in downside risk accompanied by a decrease in total risk, this theorem identifies others who would choose and those who would reject this pair of changes.

The closest existing analysis in the literature to the analysis here is Chiu (2005),[9] and a comparison between a main result in Chiu (2005) and Theorem A2 above is particularly interesting. Chiu shows that if two CDFs F(x) and G(x) satisfy

$$\begin{aligned} &G^{[2]}(b) - F^{[2]}(b) = 0 \\ &G^{[3]}(b) - F^{[3]}(b) \le 0 \end{aligned} \tag{1}$$

and the condition that there exists $z \in (a,b)$ such that

$$\begin{aligned} &G^{[3]}(x) - F^{[3]}(x) \ge 0 \;\; \mathit{for}\; x \le z \\ &G^{[2]}(x) - F^{[2]}(x) \le 0 \;\; \mathit{for}\; x \ge z \end{aligned}, \tag{2}$$

then $E_F v(x) = E_G v(x)$ implies $E_F u(x) \ge E_G u(x)$ for all u(x) with

$$-\frac{u'''(x)}{u''(x)} \ge -\frac{v'''(x)}{v''(x)} \quad \mathit{for\ all}\ x. \tag{3}$$ [10]

This result of Chiu is very similar to Theorem A2, except that Chiu's conditions are stronger, and so is his conclusion. Note first that condition (2) together with the inequality in (1) implies $G^{[3]}(x) - F^{[3]}(x) \ge G^{[3]}(b) - F^{[3]}(b)$. According to the conditions stated in Theorem A1,

[8] Eeckhoudt (2012) uses this two-step approach to construct two simple binary lotteries, in the tradition of Eeckhoudt and Schlesinger (2006), that can be ranked by the concave imprudent order (pp 148-150).
[9] Denuit and Eeckhoudt (2010b) extends Chiu (2005) from 3rd degree to general *n*th degree.
[10] We focus on the part of Chiu's (2005) Theorem 1 that is of interest to us in this paper.

therefore, Chiu's conditions are stronger than requiring G(x) to be larger than F(x) in the concave imprudent order. On the other hand, Chiu's conclusion holds for all u(x) and v(x) satisfying (3), an Arrow-Pratt type of condition for comparative 3rd degree risk aversion.[11] In contrast, the conclusion in Theorem A2 holds only for u(x) and v(x) satisfying the stronger (3/2)rd-degree Ross more risk aversion.

B. The Size-Downside Risk Tradeoff

A second tradeoff that a downside risk averse decision maker can consider is whether or not to accept an increase in downside risk when it is accompanied by an increase in the size of the random variable; that is, both an increase in downside risk and an FSD improvement occur. Again, the first change is harmful, and the second is beneficial and now it is because $u'(x) \geq 0$ is assumed. Using the same notation, consider a change from F(x) to H(x) where H(x) dominates F(x) in FSD, and a change from H(x) to G(x) where G(x) has more downside risk than H(x). Thus, the change from F(x) to G(x) involves both an increase in size and an increase in downside risk.

The question of what condition on [G(x) - F(x)] is implied by this pair of changes is addressed first. To do this, the increasing imprudent order is defined, and a theorem that characterizes this order is provided. The logic for doing this is exactly the same as that discussed in subsection A.

[11] Kimball (1990) first uses this condition for comparing the strengths of precautionary saving of different individuals.

Definition B: G(x) is larger than F(x) in the increasing imprudent order if $E_G u(x) \geq E_F u(x)$ for all u(x) with $u'(x) \geq 0$, and $u'''(x) \leq 0$.

Theorem B1: G(x) is larger than F(x) in the increasing imprudent order if and only if

$$
\begin{aligned}
&G^{[2]}(b) - F^{[2]}(b) \leq 0 \\
&G^{[3]}(x) - F^{[3]}(x) \geq G^{[3]}(b) - F^{[3]}(b), \quad \forall x \in [a,b] \\
&G^{[3]}(x) - F^{[3]}(x) \geq \left[G^{[2]}(b) - F^{[2]}(b)\right](x-a), \quad \forall x \in [a,b]
\end{aligned}
$$

The main comparative static result in this subsection is stated in the following theorem.

Theorem B2: Suppose that G(x) is larger than F(x) in the increasing imprudent order. Then

(a) $E_F v(x) \geq E_G v(x)$ implies $E_F u(x) \geq E_G u(x)$ for all u(x) who are (3/1)rd degree Ross more risk averse than v(x).

(b) $E_F v(x) \leq E_G v(x)$ implies $E_F u(x) \leq E_G u(x)$ for all u(x) who are (3/1)rd degree Ross less risk averse than v(x).

C. The SSD-Downside Risk Tradeoff

When $u'(x) \geq 0$ and $u''(x) \leq 0$ are assumed, the two primary beneficial changes that can be considered are a decrease in risk or an increase in size, and these were discussed individually in subsections A and B. Three other more complex beneficial changes can also be considered. These changes allow both size and risk to be altered, but require that the combined effect of the two changes to be beneficial. The most obvious way to do this is to both increase size and to decrease risk, and this is exactly an SSD improvement in a random variable. Thus, the next tradeoff that is analyzed involves combining a downside risk increase and an SSD improvement on the random variable. The pair of changes considered in this sub-section is from F(x) to H(x) where H(x) dominates F(x) in SSD, and then from H(x) to G(x) where G(x) has more downside risk than H(x). Thus, the total change from F(x) to G(x) involves both an SSD improvement and an increase in downside risk.

The question of what condition on F(x) and G(x) is implied by this pair of changes is answered in a manner similar to that in the previous two subsections.

<u>Definition C</u>: G(x) is larger than F(x) in the increasing concave imprudent order if $E_G u(x) \geq E_F u(x)$ for all u(x) with $u'(x) \geq 0$, $u''(x) \leq 0$ and $u'''(x) \leq 0$.[12]

<u>Theorem C1</u>: G(x) is larger than F(x) in the increasing concave imprudent order if and only if

$$\begin{aligned} &G^{[2]}(b) - F^{[2]}(b) \leq 0 \\ &G^{[3]}(x) - F^{[3]}(x) \geq G^{[3]}(b) - F^{[3]}(b), \quad \forall x \in [a,b] \end{aligned}$$

The main comparative static result in this subsection is stated in the following theorem.

<u>Theorem C2</u>: Suppose that G(x) is larger than F(x) in the increasing concave imprudent order. Then

(a) $E_F v(x) \geq E_G v(x)$ implies $E_F u(x) \geq E_G u(x)$ for all u(x) who are both (3/1)rd degree and (3/2)rd degree Ross more risk averse than v(x).

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[12] Note that throughout the paper, and for these definitions in particular, the outcome variable x is assumed to belong to a bounded interval [a, b]. While this assumption is not restrictive from an empirical point of view, it plays an important theoretical role. As pointed out by Menegatti (2014), a non-satiated individual cannot be both risk averse and imprudent on an unbounded interval $[a,\infty)$. With the assumption of a bounded domain imposed in this paper, Menegatti's cautions do not apply.

(b) $E_F v(x) \le E_G v(x)$ implies $E_F u(x) \le E_G u(x)$ for all u(x) who are both (3/1)rd degree and (3/2)rd degree Ross less risk averse than v(x).

To gain a more intuitive understanding of Theorem C2, recall that an SSD improvement from F(x) to H(x) can be broken up into two components, one part is an increase in size and the other is a decrease in risk.[13] For an SSD improvement, it can be that only one of these two components actually happens. The risk preference requirement in Theorem C2 reflects this. Because the SSD improvement could be an FSD improvement or a decrease in the risk, or both, the conditions on risk preferences used in Theorem A2 and B2 must both hold.

D. Larger in the Increasing Convex Order and Downside Risk Tradeoff: Part 1

An SSD improvement allows the possibility of an increase in size, or a decrease in risk, or both and can always be decomposed into these two separate components. Since each of the two components is beneficial, the total change, the SSD improvement, is beneficial as well. In this sub-section, the beneficial change that is considered increases both size and risk, and hence the two components of the change have opposite effects on expected utility. Defining when an increase in size and an increase in risk together are beneficial for a reference decision maker uses the increasing convex order whose definition is given in Section 2 and repeated below for convenience.

Definition 5: Random variable $\tilde{y}$ is larger than $\tilde{x}$ in the increasing convex order if $E_G[u(x)] \ge E_F[u(x)]$ for all u(x) with $u'(x) \ge 0$ and $u''(x) \ge 0$.

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[13] When H(x) dominates F(x) in SSD, there exists an intermediate CDF J(x) such that J(x) dominates F(x) in the first degree, and J(x) is riskier than H(x). $J(x) = F(x)$ for $x < s$ and $J(x) = 1$ for $x \ge 1$ serves as such an J(x). The value for s is chosen so that the mean of J(x) and the mean of H(x) are equal.

When H(x) is larger than F(x) in the increasing convex order and a reference decision maker with u'(x) ≥ 0 and u"(x) ≤ 0 prefers H(x) to F(x), then so do all decision makers who are Ross strongly less risk averse; that is, when the change from F(x) to H(x) is beneficial for some decision maker, it is also beneficial for those who are strongly less risk averse. As noted in Section 2, the increasing convex order is characterized by a condition on CDFs. H(x) is larger than F(x) in the increasing convex order if and only if $\int_a^y [H(x) - F(x)]dx \geq -Q$ for all y in [a, b] where $Q = \mu_H - \mu_F \geq 0$. The increasing convex order is extensively discussed in Liu and Meyer (2015).

The pair of changes considered in this section is from F(x) to H(x) where H(x) is larger than F(x) in the increasing convex order, and then from H(x) to G(x) where G(x) has more downside risk than H(x). The method of analysis stays the same. The conditions on F(x) and G(x) implied by this pair of changes are given in Theorem D1.

<u>Definition D</u>: G(x) is larger than F(x) in the increasing convex imprudent order if $E_G u(x) \geq E_F u(x)$ for all u(x) with u'(x) ≥ 0, u"(x) ≥ 0 and u'"(x) ≤ 0.

<u>Theorem D1</u>: G(x) is larger than F(x) in the increasing convex imprudent order if and only if

$$G^{[2]}(b) - F^{[2]}(b) \leq 0$$
$$G^{[3]}(x) - F^{[3]}(x) \geq \left[G^{[2]}(b) - F^{[2]}(b)\right](x-a), \quad \forall x \in [a,b]$$

The main comparative static result in this subsection is stated in the following theorem.

<u>Theorem D2</u>: Suppose that G(x) is larger than F(x) in the increasing convex imprudent order. Then

(a) $E_F v(x) \geq E_G v(x)$ implies $E_F u(x) \geq E_G u(x)$ for all u(x) who are both (2/1)nd degree and (3/1)rd degree Ross more risk averse than v(x).

(b) $E_F v(x) \leq E_G v(x)$ implies $E_F u(x) \leq E_G u(x)$ for all u(x) who are both (2/1)nd degree and (3/1)rd degree Ross less risk averse than v(x).

Demonstration of Theorem D2 uses Ross's strongly more risk averse definition which is the (2/1)nd degree Ross more risk averse condition, and also Definition 7 which is the (3/1)rd degree Ross more risk averse condition. These two conditions on risk preferences are involved because the change from F(x) to G(x) uses an FSD improvement to offset or compensate for both an increase in risk and an increase in downside risk. Those who are both Ross less risk averse and (3/1)rd degree Ross less risk averse view this FSD improvement even more favorably than does the reference decision maker.

E. Larger in the Increasing Convex Order and Downside Risk Tradeoff: Part 2

As in subsection D, the beneficial change considered in this section uses the increasing convex order. Rather than considering a change from F(x) to H(x) where H(x) is larger than F(x) in the increasing convex order, however, the change to H(x) is instead one where H(x) is smaller in the increasing convex order. When H(x) is smaller than F(x) in the increasing convex order and a reference decision maker with $u'(x) \geq 0$ and $u''(x) \leq 0$ prefers H(x) to F(x), then so do all decision makers who are strongly more risk averse; that is, when the change from F(x) to H(x) is beneficial for some decision maker, it is also beneficial for those who are strongly more risk averse. This change to a random variable which is smaller in the increasing convex order can also be decomposed into two components, but now the components are a decrease in size and a decrease in risk, and it is the decrease in risk that is the beneficial component of the change.

In this final subsection a decrease in size and decrease in risk together form the beneficial change. Such a change is beneficial whenever the size decrease is small enough relative to risk decrease. The pair of changes considered here is from F(x) to H(x) where H(x) is smaller than F(x) in the increasing convex order, and then from H(x) to G(x) where G(x) has more downside risk than H(x). The conditions on F(x) and G(x) implied by this pair of changes are determined in the usual manner.

Definition E: G(x) is larger than F(x) in the decreasing concave imprudent order if $E_G u(x) \geq E_F u(x)$ for all u(x) with $u'(x) \leq 0$, $u''(x) \leq 0$ and $u'''(x) \leq 0$.

Theorem E1: G(x) is larger than F(x) in the decreasing concave imprudent order if and only if

$$\begin{aligned} &G^{[2]}(b) - F^{[2]}(b) \geq 0 \\ &G^{[3]}(x) - F^{[3]}(x) + \left[G^{[2]}(b) - F^{[2]}(b)\right](b-x) \geq \left[G^{[3]}(b) - F^{[3]}(b)\right], \quad \forall x \in [a,b] \end{aligned}$$

The main comparative static result in this subsection is stated in the following theorem.

Theorem E2: Suppose that G(x) dominates F(x) in the decreasing concave imprudent order. Then

(a) $E_F v(x) \geq E_G v(x)$ implies $E_F u(x) \geq E_G u(x)$ for all u(x) who are both (2/1)nd degree Ross less risk averse and (3/2)rd degree Ross more risk averse than v(x).

(b) $E_F v(x) \leq E_G v(x)$ implies $E_F u(x) \leq E_G u(x)$ for all u(x) who are both (2/1)nd degree Ross more risk averse and (3/2)rd degree Ross less risk averse than v(x).

Demonstration of the final comparative statics theorem uses Ross's strongly more risk averse definition and Definition 6 which is the (3/2)rd degree Ross more risk averse condition. These two conditions on risk preferences are involved because the change from F(x) to G(x) uses a risk decrease to offset or compensate for both a decrease in size and an increase in downside risk. Those who are both Ross more risk averse and (3/2)rd degree Ross less risk averse view this risk decrease even more favorably than does the reference decision maker.

4. Application: Self-Protection

Ehrlich and Becker (1972) define self-protection as a costly action that reduces the probability that bad outcomes or losses occur. Dionne and Eeckhoudt (1985) and many others since then have demonstrated that comparative static analysis within this model can lead to very unusual and counterintuitive results.[14] One of the reasons for these counterintuitive findings is that self-protection always increases downside risk.

The analysis of the self-protection decision begins by decomposing the change that occurs when self-protection is increased into two components, an increase in downside risk and another change that must increase expected utility for a downside risk averse decision maker who chooses more self-protection. Depending only on the parameter values in the self-protection model, this beneficial change can be any one of the five possibilities discussed in Section 3. Thus, the self-protection decision serves to illustrate each of five tradeoffs presented in Section 3. It is also the case that the five tradeoffs in Section 3 are sufficient to cover all

[14] For example, see Briys and Schlesinger (1990), Lee (1998), Jullien, Salanie and Salanie (1999), Chiu (2000), Eeckhoudt and Gollier (2005), Liu, Rettenmaier and Saving (2009) and Meyer and Meyer (2011).

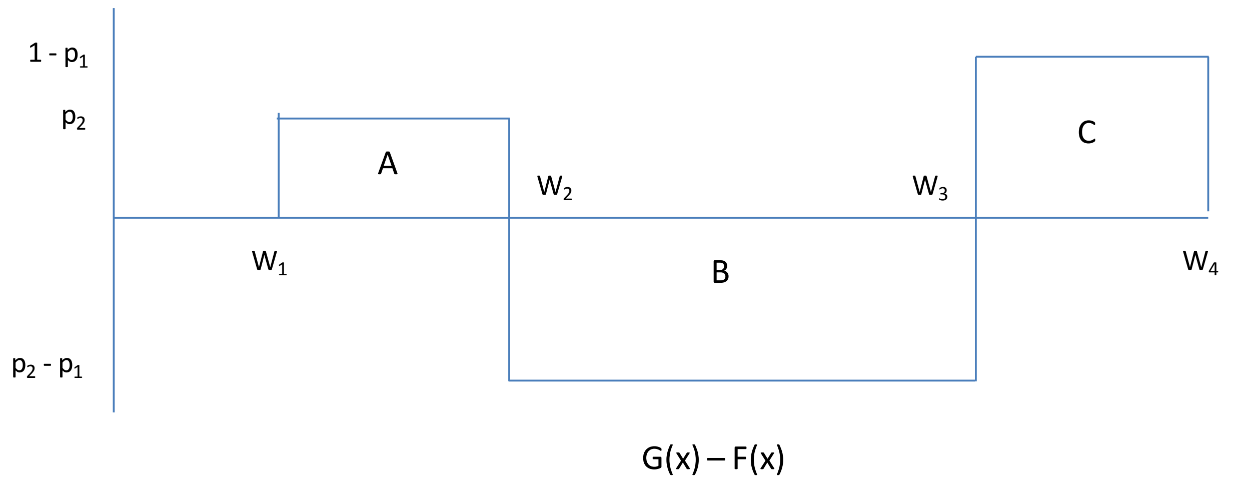
possible parameter values in the self-protection model as long as some downside risk averse decision maker chooses more self-protection.

Erhlich and Becker and many others conduct their analysis of self-protection in a simple decision model where there are just two possible outcomes, a loss of fixed size L, or no loss at all. The notation of Eeckhoudt and Gollier (2005) is used to present this model. Assume that a decision maker begins with certain wealth w and that this wealth is subject to loss $L > 0$ with probability p_1 when expenditure is e_1, and probability $p_2 < p_1$ when expenditure is $e_2 > e_1$. Final wealth W can take on one of four values listed from lowest to highest, either $W_1 = (w - L - e_2)$, $W_2 = (w - L - e_1)$, $W_3 = (w - e_2)$ and $W_4 = (w - e_1)$. [15]

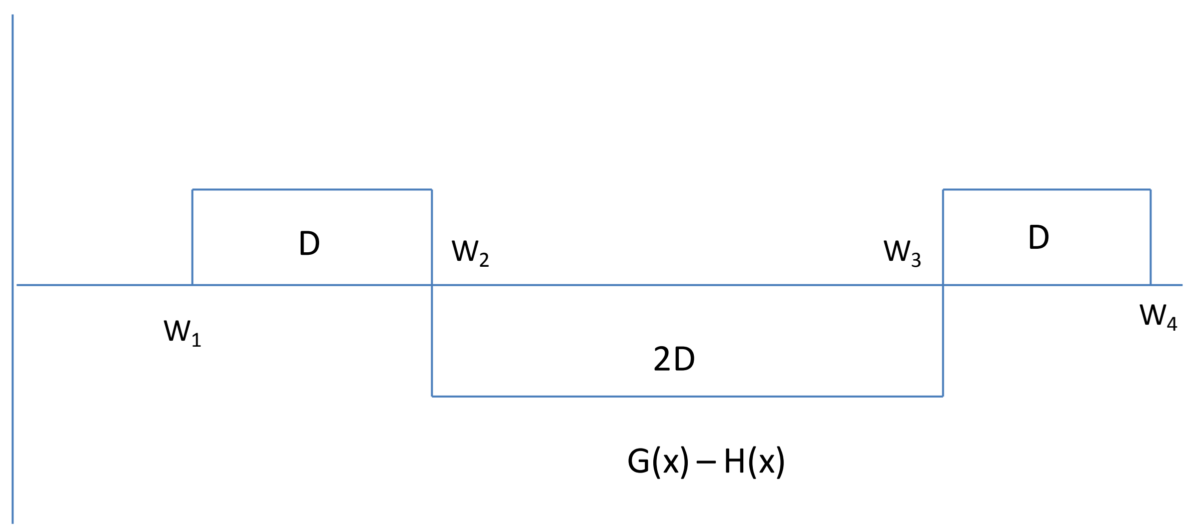
The CDF for outcome variable W associated with the higher level of self-protection e_2 is denoted G(W), and that associated with e_1 as F(W). The difference between these two CDFs, [G(W) - F(W)], is given below, and displayed graphically for a set of parameter values.

$$[G(W) - F(W)] = \begin{cases} 0 & \text{for } W < W_1 \\ p_2 & \text{for } W_1 \leq W < W_2 \\ p_2 - p_1 & \text{for } W_2 \leq W < W_3 \\ 1 - p_1 & \text{for } W_3 \leq W < W_4 \\ 0 & \text{for } W_4 \leq W \end{cases}$$

[15] It is reasonable to assume that $w - L - e_1 < w - e_2$ or equivalently that $e_2 - e_1 < L$, since under no circumstance would a rational individual expend effort on self-protection beyond the size of loss, L.

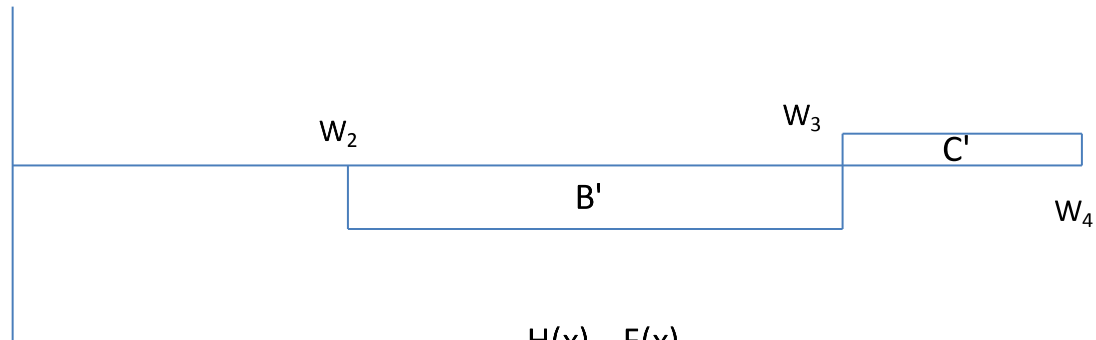


In this graph, areas A, B and C depend on the values for p_1, p_2, e_1, e_2 and L. To construct the H(x) such that G(x) is downside riskier than H(x) is a relatively simple procedure and involves the general form for [G(x) - H(x)] given below in a graph. This general form does represent an increase in downside risk.



[G(x) - H(x)] is constructed from any [G(x) - F(x)] representing more self-protection as follows. Area D is chosen to be equal to the minimum of (A, C, B/2). Subtracting this [G(x) - H(x)] from [G(x) - F(x)] yields [H(x) - F(x)]. There are only two possible cases for

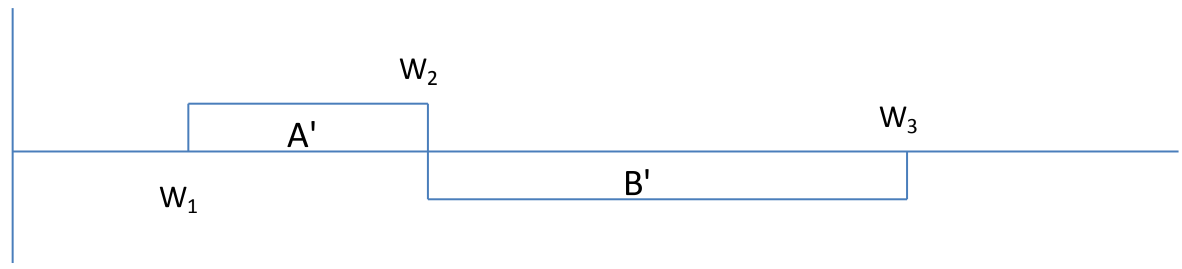
[H(x) - F(x)] to consider. For the first case it is assumed that A ≤ C and A ≤ B/2. This implies that [H(x) - F(x)] has the following graph.



H(x) – F(x)

In this graph, there are four sub-cases. If B' = C', [H(x) - F(x)] represents H(x) being less risky than F(x) so that G(x) is obtained from F(x) by an increase in downside risk accompanied by a decrease in risk and Theorem A2 applies. If B' > C', then [H(x) - F(x)] represents H(x) being an SSD increase over F(x) and Theorem C2 applies. If C' > B', and this change is expected utility increasing, then H(x) is larger than F(x) in the increasing convex order and Theorem D2 applies. Finally if area C' = 0, [H(x) - F(x)] represents H(x) being an FSD improvement over F(x) and hence Theorem B2 applies. B' = 0 is not possible because then H(x) could not be expected utility increasing relative to F(x) which is required for anyone with u'''(x) ≥ 0 to choose the higher level of self-protection.

The second case to consider is when area C < A and C < B/2. This implies that [F(x) - H(x)] has the following graph.



For this [H(x) - F(x)] to be expected utility increasing, it must be that B' > A', and it must be that H(x) is smaller than F(x) in the increasing convex order. Thus, for this increase in downside risk accompanied by a change to a smaller and less risky random variable, Theorem E2 applies. At first glance it appears that one more case is possible, when B/2 is the smaller of the three areas, but this cannot be since then the resulting [H(x) - F(x)] would represent F(x) dominating H(x) in FSD so that H(x) cannot be an expected utility increase from F(x) for any decision maker.

This completes this intuitive discussion of the self-protection example. To completely analyze the self-protection decision for decision makers who are downside risk averse, all five theorems in section 3 are used. Moreover if any downside risk averse decision maker does choose the higher level of self-protection, then one of the five theorems does predict the choices of others.

5. Conclusion

Decision making is about tradeoffs, and the size-for-risk tradeoff has been a focus in decision making under uncertainty. This paper goes beyond the traditional size-for-risk tradeoff facing risk averse decision makers and provides a framework to model tradeoffs facing downside risk averse (and risk averse) decision makers. Five new stochastic orders are introduced, each of which corresponds to a tradeoff facing downside risk averse decision makers. The CDF characterizations of these stochastic orders are provided. More importantly, it is shown that these stochastic orders, together with corresponding notions of 3rd and 2nd degree Ross more risk aversion, can be used to make predictions regarding choices of downside risk averse decision makers in environments where downside risk is a factor.

Using self-protection as an example, the paper demonstrates the applicability of the concepts and results presented here. Not only do all five stochastic orders find corresponding tradeoffs in the self-protection model, it is also the case that these five tradeoffs are the only meaningful tradeoffs that the standard self-protection model creates. The analysis confirms the findings in the literature that more downside risk averse individuals tend to invest less in self-protection, and it demonstrates this point in a more systematic and complete fashion.

Willingness to pay for reducing the probability of a loss plays an important role in project evaluation, and it is useful to know how the willingness to pay for a certain reduction in the loss probability varies with individuals' risk preferences (Dachraoui et al. 2004). Because the willingness to pay model is equivalent to the self-protection model, all results in this paper, with proper reinterpretation, apply to the interpersonal comparison of willingness to pay for a given reduction in loss probability.

References

Arrow, K. J. (1971). *Essays in the Theory of Risk-Bearing*, Markham: Chicago, IL.

Briys, E., & Schlesinger, H. (1990). Risk aversion and the propensities for self-insurance and self-protection. *Southern Economic Journal*, 57, 458–467.

Chiu, W. H. (2000). On the propensity to self-protect. *Journal of Risk and Insurance*, 67, 555–578.

Chiu, W. H. (2005). Skewness preferences, risk aversion, and the precedence relations on stochastic changes. *Management Science*, 51, 1816-1828.

Dachraoui, K., Dionne, G., Eeckhoudt, L., & Godfroid, P. (2004). Comparative mixed risk aversion: definition and application to self-protection and willingness to pay, *Journal of Risk and Uncertainty*, 29, 261-276.

Denuit, M., & Eeckhoudt, L. (2010a). Stronger measures of higher-order risk attitudes, *Journal of Economic Theory* 145, 2027-2036.

Denuit, M., & Eeckhoudt, L. (2010b). A general index of absolute risk attitude. *Management Science* 56, 712-715.

Denuit, M., Dhaene, J., Goovaerts, M., & Kaas, R. (2005). *Actuarial Theory for Dependent Risks: Measures, Orders and Models.* John Wiley & Sons: New York, NY.

Denuit, M., Lefevre, Cl., Shaked, M. (1998). The s-convex orders among real random variables, with applications. *Mathematical Inequalities and Their Applications* 1, 585-613.

Diamond, P. A., & Stiglitz, J. E. (1974). Increases in risk and in risk aversion. *Journal of Economic Theory*, 8, 337–360.

Dionne, G., & Eeckhoudt, L. (1985). Self-insurance, self-protection, and increased risk aversion. *Economics Letters*, 17. 39–42.

Eeckhoudt, L. (2012). Beyond risk aversion: Why, how and what's next? *Geneva Risk and Insurance Review*, 37, 141-155.

Eeckhoudt, L., & Gollier, C. (2005). The impact of prudence on optimal prevention. *Economic Theory*, 26, 989–994.

Eeckhoudt, L., & Schlesinger, H. (2006). Putting risk in its proper place, *American Economic Review*, 96, 280-289.

Ehrlich, I., & Becker, G. S. (1972). Market insurance, self-insurance and self-protection. *Journal of Political Economy*, 80, 623–648.

Ekern, S. (1980). Increasing Nth degree risk, *Economics Letters*, 6, 329-333.

Hadar, J., & Russell, W. (1969). Rules for ordering uncertain prospects. *American Economic Review*, 59, 25–34.

Hanoch, G., & Levy, H. (1969). The efficiency analysis of choices involving risk. *Review of Economic Studies*, 36, 335–346.

Jindapon, P., & Neilson, W. S. (2007). Higher-order generalizations of Arrow-Pratt and Ross risk aversion: A comparative statics approach. *Journal of Economic Theory*, 136, 719-728.

Jullien, B., Salanie, B., & Salanie, F. (1999). Should more risk-averse agents exert more effort? *Geneva Papers on Risk and Insurance Theory*, 24, 19-28.

Kimball, M. (1990). Precautionary saving in the small and in the large. *Econometrica*, 58, 53-73.

Lee, K. (1998). Risk aversion and self-insurance-cum-protection. *Journal of Risk and Uncertainty*, 17, 139-150.

Li, J. (2009) Comparative higher-degree Ross risk aversion, *Insurance: Mathematics and Economics* 45, 333-336.

Liu, L., Rettenmaier, A. J., & Saving, T. R. (2009). Conditional payments and self-protection. *Journal of Risk and Uncertainty*, 38, 159–72.

Liu, L., & Meyer, J. (2013). Substituting one risk increase for another: A method for measuring risk aversion. *Journal of Economic Theory*, 148, 2706-2718.

Liu, L., & Meyer, J. (2015). The increasing convex order and the tradeoff of size for risk. Working paper.

Menegatti, Mario (2014). New results on the relationship among risk aversion, prudence and temperance. *European Journal of Operational Research*, 232, 613-617.

Menezes, C., Geiss, C., & Tressler, J. (1980). Increasing downside risk. *American Economic Review*, 70, 921-932.

Meyer, D. J., & Meyer, J. (2011). A Diamond-Stiglitz approach to the demand for self-protection. *Journal of Risk and Uncertainty*, 42, 45-60.

Modica, S., & Scarsini, M. (2005). A note on comparative downside risk aversion, *Journal of Economic Theory* 122, 267-271.

Pratt, J. (1964). Risk aversion in the small and in the large. *Econometrica*, 32, 122-136.

Ross, S. A. (1981). Some stronger measures of risk aversion in the small and in the large with applications. *Econometrica*, 49, 621-663.

Rothschild, M., & Stiglitz, J. (1970). Increasing risk I: a definition. *Journal of Economic Theory*, 2, 225-243.

Shaked, M., & Shanthikumar, J. G. (2007). *Stochastic Orders.* Springer: New York, NY.

Appendix

For many proofs in this appendix, the following identity serves as a starting point, which is readily derived using integration by parts.

(*)
$$\begin{aligned} & E_F u(x) - E_G u(x) \\ & = u'(b)\left[G^{[2]}(b) - F^{[2]}(b)\right] - u''(b)\left[G^{[3]}(b) - F^{[3]}(b)\right] \\ & + \int_a^b u'''(x)\left[G^{[3]}(x) - F^{[3]}(x)\right] dx. \end{aligned}$$

A1. Proof of Theorem A1

Proof: "If" – Suppose that

(A-1)
$$\begin{aligned} & G^{[2]}(b) - F^{[2]}(b) = 0 \\ & G^{[3]}(x) - F^{[3]}(x) \geq G^{[3]}(b) - F^{[3]}(b), \quad \forall x \in [a,b] \end{aligned}$$

From (*), we have

(A-2)
$$\begin{aligned} & E_F u(x) - E_G u(x) \\ & = u'(b)\left[G^{[2]}(b) - F^{[2]}(b)\right] - u''(a)\left[G^{[3]}(b) - F^{[3]}(b)\right] \\ & + \int_a^b u'''(x)\left\{\left[G^{[3]}(x) - F^{[3]}(x)\right] - \left[G^{[3]}(b) - F^{[3]}(b)\right]\right\} dx. \end{aligned}$$

Using condition (A-1), it is readily seen that $E_G u(x) \geq E_F u(x)$ for all u(x) with u''(x) ≤ 0, and u'''(x) ≤ 0.

"Only if" – Suppose that $E_G u(x) \geq E_F u(x)$ for all u(x) with u''(x) ≤ 0, and u'''(x) ≤ 0. We need to show that (A-1) holds. First, letting u(x) = x and u(x) = - x respectively implies $G^{[2]}(b) - F^{[2]}(b) = \mu_F - \mu_G = 0$.

What remains to be shown is

(A-3)
$$G^{[3]}(x) - F^{[3]}(x) \geq G^{[3]}(b) - F^{[3]}(b), \quad \forall x \in [a,b].$$

We use proof by contradiction. Assume that (A-3) is not satisfied. That is $G^{[3]}(y)-F^{[3]}(y)<G^{[3]}(b)-F^{[3]}(b)$ for some y in [a, b]. Then, due to continuity, there exists an interval $[\alpha,\beta]\subset$(a, b) such that $G^{[3]}(y)-F^{[3]}(y)<G^{[3]}(b)-F^{[3]}(b)$ for all y in $[\alpha,\beta]$. Choose a special u(x) such that u''(a) = 0, u'''(x) < 0 for $x\in(\alpha,\beta)$ and u'''(x) = 0 otherwise. Then, from (A-2), E_Fu(x) - E_Gu(x) > 0, which contradicts that E_Gu(x) ≥ E_Fu(x) for all u(x) with u''(x) ≤ 0 and u'''(x) ≤ 0. So (A-3) must hold. Q.E.D

A2. Proof of Theorem A2

Proof: We only need to prove part (a) because part (b) can be similarly proved. We are given that G(x) is larger than F(x) in the concave imprudent order, and that E_Fv(x) ≥ E_Gv(x).

Now consider a u(x) who is (3/2)rd degree Ross more risk averse than v(x). We know that there exists $\lambda>0$ and $\phi(x)$ such that $u=\lambda v+\phi$, where $\phi''(x)\geq 0$ and $\phi'''(x)\geq 0$ for all x.

Therefore,

$$\begin{aligned}E_G u(x)&=\lambda E_G v(x)+E_G\phi(x)\leq\lambda E_F v(x)+E_G\phi(x)\\&\leq\lambda E_F v(x)+E_F\phi(x)=E_F u(x)\end{aligned},$$

where the second inequality is due to that G(x) is larger than F(x) in the concave imprudent order, and that $\phi''(x)\geq 0$ and $\phi'''(x)\geq 0$. Q.E.D.

B1. Proof of Theorem B1

Proof: "If" – Suppose that

$$\begin{aligned}&G^{[2]}(b)-F^{[2]}(b)\leq 0\\&G^{[3]}(x)-F^{[3]}(x)\geq G^{[3]}(b)-F^{[3]}(b),\quad\forall x\in[a,b]\\&G^{[3]}(x)-F^{[3]}(x)\geq\left[G^{[2]}(b)-F^{[2]}(b)\right](x-a),\quad\forall x\in[a,b]\end{aligned}\tag{B-1}$$

From Theorems C1 and D1, G(x) is larger than F(x) both in the increasing concave imprudent order and in the increasing convex imprudent order.

To show that G(x) is larger than F(x) in the increasing imprudent order, note that for any u(x) with u'(x) ≥ 0, and u'''(x) ≤ 0, there could only be the following three situations.

(i) u''(x) ≤ 0 for all x in [a, b]. Then, $E_G u(x) \geq E_F u(x)$ because G(x) is larger than F(x) in the increasing concave imprudent order;

(ii) u''(x) ≥ 0 for all x in [a, b]. Then, $E_G u(x) \geq E_F u(x)$ because G(x) is larger than F(x) in the increasing convex imprudent order;

(iii) There exists a < x* < b, such that u''(x) ≥ 0 for all x in [a, x*] and u''(x) ≤ 0 for all x in [x*, b]. Define $u_1(x)$ and $u_2(x)$ according to[16]

$$u_1'(x) \equiv \begin{cases} u'(x) & x \in [a, x^*] \\ u'(x^*) & x \in [x^*, b] \end{cases}$$

$$u_2'(x) \equiv \begin{cases} u'(x^*) & x \in [a, x^*] \\ u'(x) & x \in [x^*, b] \end{cases}.$$

Then

$$u_1''(x) = \begin{cases} u''(x) & x \in [a, x^*] \\ 0 & x \in [x^*, b] \end{cases}$$

$$u_2''(x) = \begin{cases} 0 & x \in [a, x^*] \\ u''(x) & x \in [x^*, b] \end{cases}$$

and

$$u_1'''(x) = \begin{cases} u'''(x) & x \in [a, x^*] \\ 0 & x \in [x^*, b] \end{cases}$$

$$u_2'''(x) = \begin{cases} 0 & x \in [a, x^*] \\ u'''(x) & x \in [x^*, b] \end{cases}$$

[16] Note that preferences are completely determined by the marginal utility function.

Obviously, for $u_1(x)$, $u'_1(x) \geq 0$, $u''_1(x) \geq 0$ and $u'''_1(x) \leq 0$ for all x in [a, b]. So $E_G u_1(x) \geq E_F u_1(x)$ because G(x) is larger than F(x) in the increasing convex imprudent order.

Similarly, for $u_2(x)$, $u'_2(x) \geq 0$, $u''_2(x) \leq 0$ and $u'''_2(x) \leq 0$ for all x in [a, b]. So $E_G u_2(x) \geq E_F u_2(x)$ because G(x) is larger than F(x) in the increasing concave imprudent order.

Then, noting that $u'(x) = u'_1(x) + u'_2(x) - u'(x^*)$ for all x in [a, b], that both $E_G u_1(x) \geq E_F u_1(x)$ and $E_G u_2(x) \geq E_F u_2(x)$ leads to $E_G u(x) \geq E_F u(x)$.

Summarizing (i) to (iii), for any u(x) with $u'(x) \geq 0$, and $u'''(x) \leq 0$, we have $E_G u(x) \geq E_F u(x)$. Therefore, G(x) is larger than F(x) in the increasing imprudent order.

"Only if" – Suppose that G(x) is larger than F(x) in the increasing imprudent order. Then G(x) must be larger than F(x) both in the increasing concave imprudent order and in the increasing convex imprudent order. Then from Theorems C1 and D1, it is easy to see that (B-1) holds. Q.E.D.

B2. Proof of Theorem B2

Proof: We only need to prove part (a) because part (b) can be similarly proved. We are given that G(x) is larger than F(x) in the increasing imprudent order, and that $E_F v(x) \geq E_G v(x)$.

Now consider a u(x) who is (3/1)rd degree Ross more risk averse than v(x). We know that there exists $\lambda > 0$ and $\phi(x)$ such that $u = \lambda v + \phi$, where $\phi'(x) \leq 0$ and $\phi'''(x) \geq 0$ for all x.

Therefore,

$$\begin{aligned} E_G u(x) &= \lambda E_G v(x) + E_G \phi(x) \leq \lambda E_F v(x) + E_G \phi(x) \\ &\leq \lambda E_F v(x) + E_F \phi(x) = E_F u(x) \end{aligned},$$

where the second inequality is due to that G(x) is larger than F(x) in the increasing imprudent order, and that $\phi'(x) \leq 0$ and $\phi'''(x) \geq 0$. Q.E.D.

C1. Proof of Theorem C1

Proof: "If" – Suppose that

(C-1)
$$\begin{aligned}&G^{[2]}(b)-F^{[2]}(b)\le 0\\&G^{[3]}(x)-F^{[3]}(x)\ge G^{[3]}(b)-F^{[3]}(b),\quad \forall x\in[a,b]\end{aligned}$$

From (*), we have (A-2), which is copied below for convenience.

(A-2)
$$\begin{aligned}&E_F u(x)-E_G u(x)\\&=u'(b)\left[G^{[2]}(b)-F^{[2]}(b)\right]-u''(a)\left[G^{[3]}(b)-F^{[3]}(b)\right]\\&+\int_a^b u'''(x)\left\{\left[G^{[3]}(x)-F^{[3]}(x)\right]-\left[G^{[3]}(b)-F^{[3]}(b)\right]\right\}dx.\end{aligned}$$

Using condition (C-1), it is readily seen that $E_G u(x) \ge E_F u(x)$ for all u(x) with u'(x) ≥ 0, u''(x) ≤ 0, and u'''(x) ≤ 0. That is, G(x) is larger than F(x) in the increasing concave imprudent order.

"Only if" – Suppose that $E_G u(x) \ge E_F u(x)$ for all u(x) with u'(x) ≥ 0, u''(x) ≤ 0 and u'''(x) ≤ 0. We need to show that (C-1) holds. First, letting u(x) = x implies $G^{[2]}(b)-F^{[2]}(b)=\mu_F-\mu_G\le 0$. So what remains to be shown is

(C-2)
$$G^{[3]}(x)-F^{[3]}(x)\ge G^{[3]}(b)-F^{[3]}(b),\quad \forall x\in[a,b].$$

We use proof by contradiction. Assume that $G^{[3]}(y)-F^{[3]}(y)<G^{[3]}(b)-F^{[3]}(b)$ for some y in [a, b]. Then, due to continuity, there exists an interval $[\alpha,\beta]\subset$(a, b) such that $G^{[3]}(y)-F^{[3]}(y)<G^{[3]}(b)-F^{[3]}(b)$ for all y in $[\alpha,\beta]$. Choose a special u(x) such that u'(b) = 0 u''(a) = 0, u'''(x) < 0 for $x\in(\alpha,\beta)$ and u'''(x) = 0 otherwise. Then, from (A-2) above, we have $E_G u(x) < E_F u(x)$, contradicting that $E_G u(x) \ge E_F u(x)$ for all u(x) with u'(x) ≥ 0, u''(x) ≤ 0 and u'''(x) ≤ 0. Therefore (C-2), and as a result (C-1) must hold. Q.E.D.

C2. Proof of Theorem C2

Proof: We only need to prove part (a) because part (b) can be similarly proved. We are given that G(x) is larger than F(x) in the increasing concave imprudent order, and that $E_F v(x) \ge E_G v(x)$.

Now consider a u(x) who is both (3/1)rd degree and (3/2)rd degree Ross more risk averse than v(x). By definition, there exist $\lambda_1 > 0$ and $\lambda_2 > 0$ such that $\frac{u'''(x)}{v'''(x)} \geq \lambda_1 \geq \frac{u'(y)}{v'(y)}$ and $\frac{u'''(x)}{v'''(x)} \geq \lambda_2 \geq \frac{u''(y)}{v''(y)}$ for all x and y.

Let $\lambda = \max\{\lambda_1, \lambda_2\} > 0$ and define $\phi(x)$ by $u = \lambda v + \phi$. It is easy to see that $\phi' = u' - \lambda v' \leq 0$, $\phi'' = u'' - \lambda v'' \geq 0$ and $\phi''' = u''' - \lambda v''' \geq 0$ for all x in [a, b].

Therefore,

$$\begin{aligned} E_G u(x) &= \lambda E_G v(x) + E_G \phi(x) \leq \lambda E_F v(x) + E_G \phi(x) \\ &\leq \lambda E_F v(x) + E_F \phi(x) = E_F u(x) \end{aligned},$$

where the second inequality is due to that G(x) is larger than F(x) in the increasing concave imprudent order, and that $\phi'(x) \leq 0$ $\phi''(x) \geq 0$ and $\phi'''(x) \geq 0$. Q.E.D.

D1. Proof of Theorem D1

Proof: "If" – Suppose that

$$\text{(D-1)} \qquad \begin{aligned} &G^{[2]}(b) - F^{[2]}(b) \leq 0 \\ &G^{[3]}(x) - F^{[3]}(x) \geq \left[G^{[2]}(b) - F^{[2]}(b)\right](x-a), \quad \forall x \in [a,b] \end{aligned}$$

From (*), we have

$$\text{(D-2)} \qquad \begin{aligned} &E_F u(x) - E_G u(x) \\ &= u'(a)\left[G^{[2]}(b) - F^{[2]}(b)\right] - u''(b)\left\{\left[G^{[3]}(b) - F^{[3]}(b)\right] - \left[G^{[2]}(b) - F^{[2]}(b)\right](b-a)\right\} \\ &+ \int_a^b u'''(x)\left\{\left[G^{[3]}(x) - F^{[3]}(x)\right] - \left[G^{[2]}(b) - F^{[2]}(b)\right](x-a)\right\} dx. \end{aligned}$$

Using condition (D-1), it is readily seen from (D-2) that $E_G u(x) \geq E_F u(x)$ for all u(x) with $u'(x) \geq 0$, $u''(x) \geq 0$, and $u'''(x) \leq 0$.

"Only if" – Suppose that $E_G u(x) \geq E_F u(x)$ for all u(x) with $u'(x) \geq 0$, $u''(x) \geq 0$, and $u'''(x) \leq 0$. We need to show that (D-1) holds. First, letting u(x) = x, we have

$G^{[2]}(b) - F^{[2]}(b) = \mu_F - \mu_G \leq 0$.

What remains to be shown is

(D-3) $$G^{[3]}(x) - F^{[3]}(x) \geq \left[G^{[2]}(b) - F^{[2]}(b)\right](x-a), \quad \forall x \in [a,b].$$

We use proof by contradiction. Assume that (D-3) is not satisfied. That is, $G^{[3]}(y) - F^{[3]}(y) < \left[G^{[2]}(b) - F^{[2]}(b)\right](y-a)$ for some y in [a, b]. Then, due to continuity, there exists an interval $[\alpha, \beta] \subset$ (a, b) such that $G^{[3]}(y) - F^{[3]}(y) < \left[G^{[2]}(b) - F^{[2]}(b)\right](y-a)$ for all y in $[\alpha, \beta]$. Choose a special u(x) such that u'(a) = 0, u''(b) = 0, u'''(x) < 0 for $x \in (\alpha, \beta)$ and u'''(x) = 0 otherwise. Then, from (D-2), $E_F u(x) - E_G u(x) > 0$, which contradicts that $E_G u(x) \geq E_F u(x)$ for all u(x) with $u'(x) \geq 0$, $u''(x) \geq 0$, and $u'''(x) \leq 0$. So (D-3), and hence (D-1), must hold. Q.E.D.

D2. Proof of Theorem D2

Proof: We only need to prove part (a) because part (b) can be similarly proved. We are given that G(x) is larger than F(x) in the increasing convex imprudent order, and that $E_F v(x) \geq E_G v(x)$.

Now consider a u(x) who is both (2/1)nd degree and (3/1)rd degree Ross more risk averse than v(x). By definition, there exist $\lambda_1 > 0$ and $\lambda_2 > 0$ such that $\frac{u''(x)}{v''(x)} \geq \lambda_1 \geq \frac{u'(y)}{v'(y)}$ and

$\frac{u'''(x)}{v'''(x)} \geq \lambda_2 \geq \frac{u'(y)}{v'(y)}$ for all x and y.

Let $\lambda = \min\{\lambda_1, \lambda_2\} > 0$ and define $\phi(x)$ by $u = \lambda v + \phi$. It is easy to see that $\phi' = u' - \lambda v' \le 0$, $\phi'' = u'' - \lambda v'' \le 0$ and $\phi''' = u''' - \lambda v''' \ge 0$ for all x in [a, b].

Therefore,

$$\begin{aligned} E_G u(x) &= \lambda E_G v(x) + E_G \phi(x) \le \lambda E_F v(x) + E_G \phi(x) \\ &\le \lambda E_F v(x) + E_F \phi(x) = E_F u(x) \end{aligned},$$

where the second inequality is due to that G(x) is larger than F(x) in the increasing convex imprudent order, and that $\phi'(x) \le 0$ $\phi''(x) \le 0$ and $\phi'''(x) \ge 0$. Q.E.D.

E1. Proof of Theorem E1

Proof: "If" – Suppose that

$$\text{(E-1)} \quad \begin{aligned} &G^{[2]}(b) - F^{[2]}(b) \ge 0 \\ &G^{[3]}(x) - F^{[3]}(x) + \left[G^{[2]}(b) - F^{[2]}(b)\right](b-x) \ge \left[G^{[3]}(b) - F^{[3]}(b)\right], \quad \forall x \in [a,b] \end{aligned}$$

From (*), we have

$$\text{(E-2)} \quad \begin{aligned} &E_F u(x) - E_G u(x) \\ &= u'(a)\left[G^{[2]}(b) - F^{[2]}(b)\right] - u''(a)\left\{\left[G^{[3]}(b) - F^{[3]}(b)\right] - \left[G^{[2]}(b) - F^{[2]}(b)\right](b-a)\right\} \\ &+ \int_a^b u'''(x)\left\{\left[G^{[3]}(x) - F^{[3]}(x)\right] + \left[G^{[2]}(b) - F^{[2]}(b)\right](b-x) - \left[G^{[3]}(b) - F^{[3]}(b)\right]\right\} dx. \end{aligned}$$

Using condition (E-1), it is readily seen from (E-2) that $E_G u(x) \ge E_F u(x)$ for all u(x) with u'(x) ≤ 0, u''(x) ≤ 0, and u'''(x) ≤ 0.

"Only if" – Suppose that $E_G u(x) \ge E_F u(x)$ for all u(x) with u'(x) ≤ 0, u''(x) ≤ 0, and u'''(x) ≤ 0. We need to show that (E-1) holds. First, letting u(x) = -x, we have

$$G^{[2]}(b) - F^{[2]}(b) = \mu_F - \mu_G \ge 0.$$

What remains to be shown is

(E-3) $$G^{[3]}(x)-F^{[3]}(x)+\left[G^{[2]}(b)-F^{[2]}(b)\right](b-x)\geq\left[G^{[3]}(b)-F^{[3]}(b)\right],\quad \forall x\in[a,b]$$

We use proof by contradiction. Assume that (E-3) is not satisfied. That is,

$G^{[3]}(y)-F^{[3]}(y)+\left[G^{[2]}(b)-F^{[2]}(b)\right](b-y)<\left[G^{[3]}(b)-F^{[3]}(b)\right]$ for some y in [a, b]. Then, due to continuity, there exists an interval $[\alpha,\beta]\subset$(a, b) such that

$G^{[3]}(y)-F^{[3]}(y)+\left[G^{[2]}(b)-F^{[2]}(b)\right](b-y)<\left[G^{[3]}(b)-F^{[3]}(b)\right]$ for all y in $[\alpha,\beta]$. Choose a special u(x) such that u'(a) = 0, u''(a) = 0, u'''(x) < 0 for $x\in(\alpha,\beta)$ and u'''(x) = 0 otherwise. Then, from (E-2), E_Fu(x) - E_Gu(x) > 0, which contradicts that E_Gu(x) ≥ E_Fu(x) for all u(x) with u'(x) ≤ 0, u''(x) ≤ 0, and u'''(x) ≤ 0. So (E-3), and hence (E-1), must hold. Q.E.D.

E2. Proof of Theorem E2

Proof: We only need to prove part (a) because part (b) can be similarly proved. We are given that G(x) is larger than F(x) in the decreasing concave imprudent order, and that E_Fv(x) ≥ E_Gv(x).

Now consider a u(x) who is both (2/1)nd degree Ross less risk averse and (3/2)rd degree Ross more risk averse than v(x). By definition, there exist $\lambda_1>0$ and $\lambda_2>0$ such that

$\dfrac{u''(x)}{v''(x)}\leq\lambda_1\leq\dfrac{u'(y)}{v'(y)}$ and $\dfrac{u'''(x)}{v'''(x)}\geq\lambda_2\geq\dfrac{u''(y)}{v''(y)}$ for all x and y.

Let $\lambda=\min\{\lambda_1,\lambda_2\}>0$ and define $\phi(x)$ by $u=\lambda v+\phi$. It is easy to see that

$\phi'=u'-\lambda v'\geq 0$, $\phi''=u''-\lambda v''\geq 0$ and $\phi'''=u'''-\lambda v'''\geq 0$ for all x in [a, b].

Therefore,

$$\begin{aligned}E_Gu(x)&=\lambda E_Gv(x)+E_G\phi(x)\leq\lambda E_Fv(x)+E_G\phi(x)\\&\leq\lambda E_Fv(x)+E_F\phi(x)=E_Fu(x)\end{aligned},$$

where the second inequality is due to that G(x) is larger than F(x) in the decreasing concave imprudent order, and that $\phi'(x) \geq 0$ $\phi''(x) \geq 0$ and $\phi'''(x) \geq 0$. Q.E.D.